The Classical wave eq^n

Let's consider a plane simple harmonic wave traveling through a medium along the x-axis. The wave eq^n is,

$$y = a \sin(\omega t - kx) \quad \text{--- (1)}$$

Where y is displacement of particle at point x of medium; ω - frequency, k - propagation constant

The particle velocity (u) at the point x is $\frac{dy}{dt}$ and is obtained by differentiating eq^n (1).

$$u = \frac{dy}{dt} = a\omega \cos(\omega t - kx) \quad \text{--- (2)}$$

Now, differentiating eq^n (1) w.r. to x, we have

$$\frac{dy}{dx} = -ak \cos(\omega t - kx) \quad \text{--- (3)}$$

Dividing eq (2) by (3), we get

$$\frac{u}{dy/dx} = -\frac{\omega}{k}$$

$$\boxed{\frac{\omega}{k} = \frac{2\pi\nu}{2\pi/\lambda} = \nu\lambda = v}$$

Thus the particle velocity = wave velocity x slope.

Differential eq^n of wave

Differentiating eq^n 3, we get w.r to t, keeping x-constant

$$\frac{\partial^2 y}{\partial t^2} = -a\omega^2 \sin(\omega t - kx) \quad — (5)$$

again differentiating eq. (3), we get w.r. to x keeping 't' constant.

$$\frac{\partial^2 y}{\partial x^2} = -ak^2 \sin(\omega t - kx) \quad — (6)$$

From eq^n (5) and (6), we get

$$\frac{\partial^2 y}{\partial t^2} \bigg/ \frac{\partial^2 y}{\partial x^2} = \frac{\omega^2}{k^2} = v^2$$

$$\boxed{\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}} \quad — (7)$$

This is classical wave eq^n. we can check that the same eq^n is obtained also for a wave travelling along the negative x-direc.

We may draw two important inferences from the wave eq^n (7)

(1) whether the second derivative time derivative $\frac{\partial^2 y}{\partial t^2}$ of any physical quantity related to second order-space derivative $\frac{\partial^2 y}{\partial x^2}$, a wave of sort must travel in the medium

(2) The velocity v of that wave is given by the square root of the co-efficient of second order space derivative

It must be clearly understood that the individual particle which makes up the medium do not progress through the medium with the wave; they merely oscillate (transversely or longitudinally) about their mean position.

It is their phase relationship which we observe as waves. Therefore the wave velocity is also called phase velocity. It is the velocity with which planes of equal phase (crest or troughs) in the case of transverse wave and compression or rarefraction in case of longitudnal wave) travels ~~with~~ thorough the medium

The phase velocity of a wave is given by

$$v = \nu\lambda$$
$$= 2\pi\nu \times \frac{\lambda}{2\pi} \quad , v = \frac{\omega}{k} \quad \text{--- (8)}$$

The relation may also be obtained as follows: (4)
cf (8)

The total phase $\phi(x,t)$ of a harmonic wave is defined as the argument of the sine (or cosine) function. The phase of wave travelling in $+x$-direction is

$$\phi(x,t) = \omega t - kx$$

At a given space point x, the phase increases linearly with time (as is indicated by term ωt) At a given t, the phase decreases linearly with x (as is indicated by the term $-kx$) The decrease in phase is due to the fact that at greater x the phase corresponds to waves emitted at earlier time.

Let us now calculate the velocity with which, for instance, a wave crest i.e. maximum value of $\sin(\omega t - kx)$ or $\sin\phi$ moves. This is the wave velocity.

Suppose we have a crest at x at time t and it appears at $x+dx$ at a later time $(t+dt)$. To keep the phase constant, the total derivative of phase must be zero. now $d\phi$ is given by

$$d\phi = \left(\frac{\partial\phi}{\partial t}\right)dt + \left(\frac{\partial\phi}{\partial x}\right)dx$$