**DDU GORAKHPUR UNIVERSITY, GORAKHPUR, UTTAR PRADESH**

**Department of Physics**

**Syllabus and Units Cover by Dr. Sintu Kumar**

**Semester I**

**Paper I: Mathematical Physics (Unit – II)**

**Course: M.Sc. I**

**Fourier Transform:** Dirac Delta function, Fourier Transform, Sine and Cosine transform, Linearity, Change of Scale, Translation, Modulation, simple applications.

**Green Function:** Green's function as a technique to solve linear ordinary differential equations, Homogeneous and Inhomogeneous boundary conditions, Solution of Poisson equation using Green's function technique, Symmetry property.

**Reading Books**

1. Advanced Engineering Mathematics by Erwin Kreyszig
2. Mathematical Physics by H.K. Das

**References for uploaded lectures**

1. Advanced Engineering Mathematics by Erwin Kreyszig
2. Mathematical Physics by H.K. Das
3. NPTEL IIT Kharagpur
4. MIT Opencourseware

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# Fourier Transform

## Fourier integral representation

Letting $L \to \infty$ in a fourier series lead to the introduction of a different type of representation called a Fourier integral representation where the function $f(x)$ is defined $\forall x$ and need not to be periodic

This representation forms the basis of an integral transform called the Fourier transform that is similar to Laplace transform

Let a function $f(x)$ be defined in an infinite interval $(-\infty, \infty)$ and absolutely integrable it i.e. $\exists$ an integral

$$\int_{-\infty}^{\infty} |f(x)|\, dx = Q \quad \text{—} ①$$

Fourier series in the interval $(-L, L)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \quad \text{—} ②$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(t)\, dt$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(t) \cos\frac{n\pi}{L} t\, dt$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(t) \sin\frac{n\pi}{L} t\, dt$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt + \frac{1}{L}\sum_{n=1}^{\infty} \left(\int_{-L}^{L} f(t) \cos\frac{n\pi}{L} t\, dt\right) \cos\frac{n\pi}{L} x$$
$$+ \frac{1}{L}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(t) \sin\frac{n\pi}{L} t\, dt\right) \cdot \sin\frac{n\pi}{L} x$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t)\left[\cos\frac{n\pi}{L} t \cos\frac{n\pi x}{L} + \sin\frac{n\pi}{L} t \sin\frac{n\pi x}{L}\right]$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\, dt + \frac{1}{L}\sum_{n=1}^{\infty}\int_{-L}^{L} f(t) \cos\frac{n\pi (t-x)}{L}\, dt \quad \text{—} ③$$

when $L \to \infty$

$$\alpha_1 = \frac{\pi}{L},\ \alpha_2 = \frac{2\pi}{L} \cdots \alpha_n = \frac{n\pi}{L},\ \alpha_{n+1} = \frac{(n+1)\pi}{L}$$

$$\alpha_{n+1} - \alpha_n = \frac{\pi}{L} \text{ i.e. } \Delta\alpha_n = \frac{\pi}{L} \quad \text{—} (4)$$

$$f(x) = \frac{1}{2L}\int_{-L}^{L} f(t)\,dt + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\int_{-L}^{L} f(t)\cos\alpha_n(t-x)\,dt\right)\Delta\alpha_n$$

when $L \to \infty$

$\therefore \cos\frac{n\pi}{L}(t-x)$

I. $$\left|\frac{1}{2L}\int_{-L}^{L} f(t)\,dt\right| \le \frac{1}{2L}\int_{-L}^{L} f(t)\,dt$$

$$< \frac{1}{2L}\int_{-\infty}^{\infty} |f(t)|\,dt$$

* For fixed $L$, the expression in the parenthesis is a function of $\alpha_n$ which takes on values from $\frac{\pi}{L}$ to $\infty$

$$= \frac{1}{2L}\,Q \to 0$$

whenever $L \to \infty$

II. For fix $L$, $\alpha_n \to \frac{\pi}{L}$ to $\infty$
as $L \to \infty$

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha_n(t-x)\,dt\right)d\alpha_n \quad \text{—} (6)$$

Note
* FIR of $f(x)$ at a point if $f(x)$ is continuous.
* Suppose at a particular point if $f(x)$ is discontinuous

$$\frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha_n(t-x)\,dt\right)d\alpha_n = \frac{f(x+0)+f(x-0)}{2}$$

$\because \cos\alpha(t-x) = \cos\alpha t\cos\alpha x + \sin\alpha t\sin\alpha x$

Now,

$$f(x) = \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\cos\alpha t\,dt\right)\cos\alpha x\,d\alpha + \frac{1}{\pi}\int_{0}^{\infty}\left(\int_{-\infty}^{\infty} f(t)\sin\alpha t\,dt\right)\sin\alpha x\,d\alpha. \quad \text{—} (7)$$