Proof (ii)

$$F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos ax \cdot \sin \alpha x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cdot\frac{1}{2}\left\{\sin(\alpha+a)x + \sin(\alpha-a)x\right\}dx$$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\alpha+a)x\,dx$$

$$+ \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(\alpha-a)x\,dx$$

$$= \frac{1}{2}\left[F_s(\alpha+a) + F(\alpha-a)\right]$$

R.H.S

Similarly (iii), (iv) and (v) can be proved

Example :-

Find Fourier transform of $\cos\frac{x^2}{2}$ and $\sin\frac{x^2}{2}$

Solution :- we already evaluated,

$$F[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}}e^{-\alpha^2/4a^2}$$

Let $a = \frac{1}{\sqrt{2}}e^{-i\pi/4}$

$$= \frac{1}{\sqrt{2}}\cos\left(\pi/4 - i\sin\pi/4\right) = \frac{1}{2}(1-i)$$

$$a^2 = \frac{1}{2}e^{-i\pi/2}$$

if I put $a^2 = \frac{1}{2}\left(\cos\frac{\pi}{2} - i\sin\pi/2\right) = -i/2$

$$F[e^{ix^2/2}] = \frac{1}{\sqrt{2}\,\frac{1}{2}(1-i)}e^{+\alpha^2/4\cdot i/2}$$

$$= \frac{\sqrt{2}}{1-i}e^{+\alpha^2/2i}$$

$$= \frac{\sqrt{2}}{1-i}e^{i\alpha^2/2} \checkmark = \frac{\sqrt{2}(1+i)}{2}e^{-i\alpha^2/2}$$

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$$\mathcal{F}[e^{ix^2/2}] = \frac{1+i}{\sqrt{2}} e^{-i\alpha^2/2}$$

$$\mathcal{F}[\cos\frac{x^2}{2}] + i\mathcal{F}[\sin x^2/2] = \frac{1+i}{\sqrt{2}}[\cos\alpha^2/2 - i\sin\alpha^2/2]$$

taking real and imaginary part,

$$= \frac{1}{\sqrt{2}}\cos\frac{\alpha^2}{2} - \frac{i}{\sqrt{2}}\sin\alpha^2/2 + \frac{i}{\sqrt{2}}\cos\alpha^2/2 + \frac{1}{\sqrt{2}}\sin\alpha^2/2$$

$$\frac{1}{\sqrt{2}}(\cos\alpha^2/2 + \sin\alpha^2/2) + \frac{i}{\sqrt{2}}(\cos\alpha^2/2 - \sin\alpha^2/2)$$

Equating,

$$\mathcal{F}[\cos x^2/2] = \frac{1}{\sqrt{2}}(\cos\alpha^2/2 + \sin\alpha^2/2)$$

$$\mathcal{F}[\sin x^2/2] = \frac{1}{\sqrt{2}}(\cos\alpha^2/2 - \sin\alpha^2/2)$$

Example :-

Evaluate Fourier transform of $e^{-4x^2}\cos 4x$

Solution

$$\mathcal{F}[f(x)\cos ax] = \frac{1}{2}[F(\alpha+a) + F(\alpha-a)]$$

$$\mathcal{F}[e^{-4x^2}\cos 4x] =$$

we know that,

$$\mathcal{F}[e^{-a^2x^2}] = \frac{1}{a\sqrt{2}} e^{-\alpha^2/4a^2}$$

$$\mathcal{F}[e^{-4x^2}] = \frac{1}{2\sqrt{2}} e^{-\alpha^2/16}$$

$$\mathcal{F}[e^{-4x^2}\cos 4x] = \frac{1}{2}\left[\frac{1}{2\sqrt{2}} e^{-\left(\frac{\alpha+4}{4}\right)^2} + \frac{1}{2\sqrt{2}} e^{-\left(\frac{\alpha-4}{4}\right)^2}\right]$$

$$= \frac{1}{4\sqrt{2}}\left[e^{-(\alpha+4/4)^2} + e^{-(\alpha-4/4)^2}\right]$$

⑤ Fourier transform of Derivate and integral of a function (8)

Theorem

(i) $\mathcal{F}[x^n f(x)] = (-1)^n \frac{d^n}{d\alpha^n} F(\alpha)$

(ii) $\mathcal{F}_s[x f(x)] = -\frac{d}{d\alpha} F_c(\alpha)$

(iii) $\mathcal{F}_c[x f(x)] = \frac{d}{d\alpha} F_s(\alpha)$

Proof

(i) $F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$

$$\frac{d^n}{d\alpha^n} F(\alpha) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) (ix)^n e^{i\alpha x} dx$$

$$= \frac{i^n}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) x^n e^{i\alpha x} dx$$

$$= i^n \mathcal{F}[f(x) \cdot x^n]$$

Therefore,

$$\mathcal{F}[f(x) x^n] = \frac{1}{i^n} \frac{d^n}{d\alpha^n} F(\alpha)$$

$$\boxed{\mathcal{F}[f(x) x^n] = (-i)^n \frac{d^n}{d\alpha^n} F(\alpha)}$$

Proof (ii)

$$\mathcal{F}_c[\alpha] = \sqrt{2/\pi} \int_0^{\infty} f(x) \cos \alpha x \, dx$$

$$\frac{d}{d\alpha} F_c(\alpha) = \sqrt{2/\pi} \int_0^{\infty} f(x) \frac{d}{d\alpha}(\cos \alpha x) \, dx$$

$$= -\sqrt{\frac{2}{\pi}} \int_0^{\infty} x f(x) \sin \alpha x \, dx$$

$$= -F_s[x f(x)]$$

$$\boxed{F_s[x f(x)] = -\frac{d}{d\alpha} F_c(\alpha)}$$

Similarly (iii) can be prooved