Properties of Fourier transform

① Theorem [Linearity Property]

Fourier transform, Fourier - sine transform, and Fourier cosine transform are linear i.e.

(i) $\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$

(ii) $\mathcal{F}_s[af(x) + bg(x)] = a\mathcal{F}_s[f(x)] + b\mathcal{F}_s[g(x)]$

(iii) $\mathcal{F}_c[af(x) + bg(x)] = a\mathcal{F}_c[f(x)] + b\mathcal{F}_c[g(x)]$

Where a, b are real number,

Proof

(i) $$\mathcal{F}[af(x) + bg(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)] e^{i\alpha x} dx$$

$$= \frac{a}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx + \frac{b}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{i\alpha x} dx \quad \text{— ①}$$

We know that,

$$\mathcal{F}[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\alpha x} dx$$

Now from Eq^n ①, we get

$$= a\mathcal{F}[f(x)] + b\mathcal{F}[g(x)]$$

Similarly we can proove the (ii) & (iii) theorem for sine and cosine.

② Shifting theorem

Theorem

$$\mathcal{F}[f(x-a)] = e^{i\alpha a} F(\alpha), \text{ where } F(\alpha) = \mathcal{F}[f(x)]$$

Proof:- Starting from L.H.S.

$$\mathcal{F}[f(x-a)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a) e^{i\alpha x} dx$$

put : $t = x - a$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\alpha(a+t)}\, dt$$

$$= \frac{e^{i\alpha a}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{i\alpha t}\, dt$$

$$= e^{i\alpha a}\, \mathcal{F}[f(t)] = F(\alpha)\, e^{i\alpha a}$$

③ Theorem

$$\mathcal{F}[e^{iax} f(x)] = F(\alpha + a)$$

Proof :- start from L.H.S.

$$\mathcal{F}[e^{iax} f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{iax} f(x) . e^{i\alpha x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{i(\alpha+a)}\, dx$$

$$= F(\alpha + a) \quad \text{R.H.S.}$$

④ Change of scale property

Theorem For any non zero real number a

(i) $\mathcal{F}[f(x)] = \frac{1}{|a|} F(\alpha/a)$

(ii) $\mathcal{F}_s[f(ax)] = \frac{1}{a} F_s(\alpha/a)\ ;\ a>0$

(iii) $\mathcal{F}_c[f(x)] = \frac{1}{a} F_c(\alpha/a)\ ;\ a>0$

Proof :- Let, a>0,

$$\mathcal{F}[f(ax)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax)\, e^{i\alpha x}\, dx$$

put $t = ax$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cancel{f(a/t)}\ f(t)\, e^{i\alpha t/a}\, dt/a$$

$$= \frac{1}{a} F(\alpha/a) \quad \text{R.H.S.}$$

(ii) Let $a<0$

$$\mathcal{F}[f(ax)] = -\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\, e^{i\alpha t/a}\frac{dt}{a}$$

$$= -\frac{1}{a} F\left(\alpha/a\right)$$

$$\therefore \boxed{\mathcal{F}[f(ax)] = \frac{1}{|a|} F\left(\alpha/a\right)}$$

④ Modulation Property

Theorem

(i) $\mathcal{F}[f(x)\cos ax] = \frac{1}{2}[F(\alpha+a) + F(\alpha-a)]$

(ii) $\mathcal{F}_s[F(x)\cos ax] = \frac{1}{2}[F_s(\alpha+a) + F_s(\alpha-a)]$

(iii) $\mathcal{F}_s[f(x)\sin ax] = \frac{1}{2}[F_c(\alpha-a) - F_c(\alpha+a)]$

(iv) $\mathcal{F}_c[f(x)\sin ax] = \frac{1}{2}[F_s(a+\alpha) + F_s(a-\alpha)]$

(v) $\mathcal{F}_c[f(x)\cos ax] = \frac{1}{2}[F_c(\alpha+a) + F_c(\alpha-a)]$

Proof :- starting from L.H.S.

$$\mathcal{F}[f(x)\cos ax] = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\cos ax\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\alpha x} f(x)\left[\frac{e^{iax} + e^{-iax}}{2}\right] dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(\alpha+a)x} f(x)\, dx\right.$$

$$\left. + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(\alpha-a)x\, dx} f(x)\, dx\right]$$

$$\mathcal{F}[f(x)\cos ax] = \frac{1}{2}[F(\alpha+a) + F(\alpha-a)]$$ proved

R.H.S.