Applications of Linear Programming

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All organizations, big or small, have at their disposal, men, machines, money and materials, the supply of which may be limited. If the supply of these resources are unlimited, the need for management tools like linear programming will not arise at all. Availability of resources being limited, the management must find the best allocation of its resources in order to maximize the profit or minimize the loss or utilize the production capacity to the maximum extent. However, this involves a number of problems which can be overcome by quantitative methods particularly by the linear programming technique.

Generally speaking, linear programming can be used for optimization problems if the following conditions are satisfied :

(i) There must be a well defined objective function such as profit, cost or quantities produced which is to be either maximized or minimized and which can be expressed as a linear function of decision variables.

(ii) There must be constraints on the amount or extent of attainment of the objective and these constraints must be capable of being expressed as linear equations and/or linear inequalities in terms of decision variables.

(iii) There must be alternative courses of action. For example, a given product may be processed by two different machines and problem may be as to how much of the product to allocate to each machine.

(iv) Another necessary requirement is that the decision variables should be inter-related and non-negative. The non-negativity conditions show that linear programming deals with real life situations for which negative quantities are generally illogical.

(i) As stated earlier that the resources must be in limited supply. For example if a firm starts producing greater number of a particular product, then it must make smaller number of other products, as the total production capacity is limited.

Assumptions in Linear Programming Problem

As experienced and seen from above examples, that the assumptions in linear programming problem that limit its applicability are as follows :

(a) Proportionality: A primary requirement of linear programming problem is that the objective function and every constraint function must be ***linear.*** Linearity implies that the product of variables such as x_1x_2, powers of variables such as , and combination of variables such as $a_1x_1 + a_2 \log x_2$, are not allowed.

(b) Additively : Additively means if it takes t_1 hours on machine M to make product A and t_2 hours to make product B, then the time on machine M devoted to produce A and B both is $t_1 + t_2$, provided the time required to change the machine from product A to B is negligible.

(c) Multiplicatively: Multiplicatively means if it takes one hour to make a single item on a given machine, then it will take 10 hours to make 10 such items and the total profit from selling a given number of units is the unit profit times the number of units sold.

(d) Divisibility or Continuity : It means that the fractional levels must be permissible besides integral values of decision variables and available resources.

(e) Deterministic : All the parameters in the linear programming models are assumed to be known exactly.

A practical problem which completely satisfies all the above assumptions for linear programming is very rare indeed. Therefore, the user should be fully aware of the assumptions and approximations involved and should satisfy himself that they are justified before proceeding to apply linear programming technique.

Limitations of Linear Programming

In spite of wide area of applications, some limitations are associated with linear programming techniques. These limitations are as follows :

1. In some problems, objective function and constraints are not linear. Generally, in real life situations concerning business and industrial problems, constraints are not linearly treated to variables.

2. There is no guarantee of getting integer valued solutions, for example, in finding out how many men and machines would be required to perform a particular job, rounding off the solution to the nearest integer will not give an optimal solution. Integer programming deals with such problems instead of linear programming.

3. Linear programming model does not take into consideration the effect of time and uncertainty. Thus, the model should be defined in such a way that any change due to internal as well as external factors can be incorporated.

4. Sometimes large-scale problems cannot be solved with linear programming techniques even when the computer facility is available. Such difficulty may be removed by decomposing the main problem into several small problems and then solving them separately.

5. Parameters appearing in the model are assumed to be constant and deterministic. But, in real life situations, they are neither constant nor deterministic.

6. Linear programming deals with only single objective, whereas in real-life situations, we may come across with conflicting multi objectives problems. ***Goal programming*** and ***multi-objective programming*** deal with such problems instead of linear programming.

Applications of Linear Programming

Linear programming is the most widely used technique in business organizations, industries and in so many other fields of allotment of resources such as men, materials, machines, money, etc. including the management decision problems. Some important applications of linear programming in our life are as follows :

1. Assignment Problems : Linear programming techniques may be used in assignment problems where the available resources that may be in the form of manpower, machines, materials, money, etc. are to be alloted or assigned (i.e. allocated) to various performances (jobs) in the best manner that minimizes the total cost or maximizes the total profit (i.e. optimizes the performance).

2. Transportation Problems : Linear programming techniques may be used in transportation problems in which we require the transportation (shipping) of available amounts from the sources (origins) to the destinations according to their requirements so that the transportation cost is minimum.

3. Blending Problems : Linear programming techniques may be used in blending problems in which we decide the quantities of various available components of the product for mixing to produce a new product so that the total cost of the product is minimized which in turn maximize the total profit.

5. Agricultural Problems : Linear programming techniques may be used in agricultural planning for allocating the limited resources such as labourer, water-supply, fertilizer, pesticides and working capital, etc. so as to maximize the net revenue.

6. Military Problems : Linear programming techniques may be applied by military commanders (officers) in selecting the number of defence units, design of weapons, optimal bombing patterns according to their availability so that the outcome of their use is in the best interest of the country.

7. Production Management Problems : Linear programming techniques may be used by production management for determining product mix, product smoothing, and assembly time-balancing by using the available inventories considering the production capacity, man powers, various costs involved in production, constraints on production, etc. so as to maximize the total profit.

8. Marketing Management Problems: Linear programming helps in analysing the effectiveness of advertising campaign and time based on the available advertising media. It also helps travelling sales-man in finding the shortest route for his tour.

9. Manpower Management Problems : Linear programming allows the personnel manager in a firm to analyse personnel policy combinations in terms of their appropriateness for maintaining a steady-state flow of people into through and out of the firm.

10. Physical Distribution Problems : Linear programming determines the most economic and efficient manner of locating manufacturing plants and distribution centres for physical distribution.

Besides above, linear programming involves the applications in the area of administration, education, inventory control, awarding contract, capital budgeting, etc. Indian Railways and other railways are also using the techniques of linear programming in the selection of routes and allocation of trains to various routes as well in reservation of tickets. Indian Air-Lines and other air-lines are also using the techniques of linear programming in the selection of routes and allocations of air-crafts to various routes.

In fact, linear programming techniques may be used in any situation, where an objective function expressed as a linear function is to be optimized (either maximized or minimized) subject to certain constraints expressed as linear equations and/or linear inequalities.

Advantages of Linear Programming Techniques

The advantages of linear programming techniques may be out-lined as follows :

1. Linear programming technique helps us in making the optimum utilization of productive resources. It also indicates how a decision maker can employ his productive factors most effectively by choosing and allocating these resources.

2. The quality of decisions may also be improved by linear programming technique. The user of this technique becomes more objective and less subjective.

3. Linear programming technique provides practically applicable solutions since there might be other constraints operating outside the problem which must also be taken into consideration just because, so many units must be produced does not mean that all those can be sold. So the necessary modification of its mathematical solution is required for the sake of convenience to the decision maker.

4. In production processes, high-lighting of bottlenecks is the most significant advantage of this technique. For example, when bottlenecks occur, some machines cannot meet the demand while others remain idle for some time.

5. The necessary modification of the mathematical solutions is also possible by using linear programming.

6. Linear programming helps in re-evaluation of a basic plan with the changing conditions.

Slack and Surplus Variables in Linear Programming Problem

The general linear programming problem can be stated as follows :

Find the values of n decision variables $x_1, x_2, ..., x_n$ so as to optimize

i.e. maximize or minimize $Z = c_1x_1 + c_2x_2 + ... + c_j x_j + ... + c_n x_n$ ...(6)

subject to m-constraints :

$$\left.\begin{array}{l} a_{11}x_1 + a_{12} x_2 + ... + a_{1j} x_j + ... + a_{1n} x_n \ (\leq \text{ or } = \text{ or } \geq)\ b_1 \\ a_{21}x_1 + a_{22} x_2 + ... + a_{2j} x_j + ... + a_{2n} x_n \ (\leq \text{ or } = \text{ or } \geq)\ b_2 \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{i1}x_1 + a_{i2} x_2 + ... + a_{ij}x_j + ... + a_{in} x_j \ (\leq \text{ or } = \text{ or } \geq)\ b_i \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1 + a_{m2} x_2 + ... + a_{mj}x_j + ... + a_{mn}x_n \ (\leq \text{ or } = \text{ or } \geq)\ b_m \end{array}\right\} \quad ...(7)$$

where constraints may be in the form of any inequality ($\leq$ or $\geq$) or even in the form of an equation (=), and finally satisfying non-negativity restrictions :

$$x_1 \geq 0, x_2 \geq 0, ..., x_j \geq 0, ... x_n \geq 0 \quad ...(8)$$

where all $c_1, c_2, ..., c_n$; $a_{11}, a_{12}, ..., a_{1n}, a_{21}, a_{22}, ..., a_{2n}, ..., a_{m1}, a_{m2}, ..., a_{mn}$; $b_1, b_2, ..., b_m$ are known constants. However, by convention, the values of right hand side parameters $b_1, b_2, ..., b_m$ are restricted to non-negative values only. It is important to note that any negative b_i can be changed to a positive value on multiplying both sides of the constraint by – 1. This will not only change the sign of all left side coefficients and right side parameters but will also change the direction of the inequality sign.

In sigma (Σ) notation, the above general linear programming problem can be written as follows :

Find the values of the n-decision variables x_j, $j = 1, 2, ..., n$ so as to

Optimize (either maximize or minimize) $Z = \sum_{j=1}^{n} c_j x_j$ ...(9)

subject to m-constraints : $\sum_{j=1}^{n} a_{ij} x_j$ ($\leq$ or = or $\geq$) b_i ; $i = 1, 2, ..., m$. ...(10)

and satisfying : $x_j \geq 0, j = 1, 2, ..., n.$ ...(11)

The inequality constraints of a general linear programming problem may be changed to equalities as and when required, by adding non-negative variables to the left hand side of $\leq$ type of inequality constraints and by subtracting non-negative variables from the left hand side of $\geq$ type of inequality constraints. Thus, we arrive at the following definitions :

Slack Variables:

If some constraints of a general linear programming problem are of the type

$$\sum_{j=1}^{n} a_{ij}\, x_i \leq b_i;\ i = 1, 2, ..., k. \qquad ...(12)$$

then, the non-negative variables 's_i' to be added to left hand sides of such constraints which satisfy

$$\sum_{j=1}^{n} a_{ij}\, x_i + s_i = b_i;\ i = 1, 2, ..., k. \qquad ...(13)$$

are called ***slack variables.***

For example, the inequality constraints $x_1 + x_2 \leq 3$ and $2x_1 + 3x_2 \leq 4$, where $x_1 \geq 0$, $x_2 \geq 0$ may be changed to equalities by adding the slack variables $s_1 \geq 0$ and $s_2 \geq 0$ to the left hand sides of above inequalities respectively. Thus, by introducing slack variables s_1 and s_2, we get the following equalities :

$x_1 + x_2 + s_1 = 3$ and $2x_1 + 3x_2 + s_2 = 4$, where $x_1 \geq 0$, $x_2 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$.

Surplus Variables: If some constraints of a general linear programming problem are of the type

$$\sum_{j=1}^{n} a_{ij}\, x_i \geq b_i;\ i = k, k + 1, ..., m. \qquad ...(14)$$

then, the non-negative variables 's_i' to be subtracted from left hand sides of such constraints which satisfy

$$\sum_{j=1}^{n} a_{ij}\, x_i - s_i = b_i;\ i = k, k + 1, ..., m. \qquad ...(15)$$

are called ***surplus variables.***

For example, the inequality constraints $3x_1 + 4x_2 \geq 5$ and $5x_1 + 6x_2 \geq 8$, where $x_1 \geq 0$, $x_2 \geq 0$ may be changed to equalities by subtracting the surplus variables $s_3 \geq 0$ and $s_4 \geq 0$ from the left hand sides of the above inequalities respectively. Thus, by introducing surplus variables s_3 and s_4, we get the following equalities :

$3x_1 + 4x_2 - s_3 = 5$ and $5x_1 + 6x_2 - s_4 = 8$, where $x_1 \geq 0$, $x_2 \geq 0$, $s_3 \geq 0$, $s_4 \geq 0$.

Matrix Notation of General Linear Programming Problem

Consider the following general linear programming problem:

Find the values of n decision variables $x_1, x_2, ..., x_n$ so as to optimize

i.e. maximize or minimize Z = $\sum_{j=1}^{n} c_j \, x_j$

subject to m-constraints : $\sum_{j=1}^{n} a_{ij} \, x_i$ ($\leq$ or = or $\geq$) b_i; $i = 1, 2, ..., m$

and satisfying the non-negativity restrictions : $x_j \geq 0$, $j = 1, 2, ..., n$.

The above general linear programming problem in matrix notation can be written as :

Find the column vector X so as to optimize **Z = CX** …(16)

subject to the constraints **AX** ($\leq$ or = or $\geq$) **b** ...(17)

and satisfying **X≥0** ...(18)

where **C** = $[c_1, c_2, ..., c_n]$, a row vector

$$\mathbf{X} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ a column vector}$$

$\mathbf{A} = [a_{ij}]$, a matrix of coefficients a_{ij} where $i = 1, 2, ..., m$ and $j = 1, 2, ..., n$

and $$\boldsymbol{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \text{ a column vector.}$$

Canonical and Standard Forms of Linear Programming Problem

When a problem given in textual form or otherwise, is to be solved by linear programming technique, then it is first essential to write it in the model form (i.e. a mathematical form) to have effective use and application of linear programming technique :

After the model formulation of the problem (in LPP model), the next step is to obtain its solution. But, before any method is used to find its solution, the linear programming problem must be presented in a suitable form. As such, we explain its following two forms :

Canonical Form: A general linear programming problem (GLPP) can always be put in the following form :

Find the values of n-decision variables x_j; $j = 1, 2, ..., n$ so as to

Maximize $Z = c_1 x_1 + c_2 x_2 + ... + c_n x_n$ ...(19)

subject to m-constraints : $\sum_{j=1}^{n} a_{ij} x_i \leq b_i$; $i = 1, 2, ..., m$ ...(20)

and satisfying the non-negativity restrictions : $x_j \geq 0, j = 1, 2, ..., n$ ...(21)

by making some elementary transformations. This form of the LPP is called its ***maximization canonical form*** and has the following characteristics :

(i) Objective function is of maximization type,

(ii) All constraints are of less than or equal $\leq$ type,

(iii) All decision variables x_j, $j = 1, 2, ..., n$ are non-negative.

This canonical form is a format for a linear programming problem, which finds its use in duality theory.

The general linear programming problem can also be put in its ***minimization canonical form*** which is given below :

Find the values of n-decision variables x_j; $j = 1, 2, ..., n$ so as to

Minimize $Z = \sum_{j=1}^{n} c_j x_j = c_1 x_1 + c_2 x_2 + ... + c_n x_n$ ...(22)

subject to m-constraints : $\sum_{j=1}^{n} a_{ij} x_i \geq b_i$; $i = 1, 2, ..., m$ ...(23)

and satisfying the non-negativity restrictions : $x_j \geq 0$; $j = 1, 2, ..., n$ ...(24)

This canonical form has the following characteristics :

(i) Objective function is of minimization type,

(ii) All constraints are of more than or equal $\geq$ type,

(iii) All decision variables x_j; $j = 1, 2, ..., n$ are non-negative.

Standard Form: A general linear programming problem (GLPP) can also be put in the following form :

Find the values of n-decision variables x_j; $j = 1, 2, ..., n$ so as to optimize

i.e. maximize and minimize $Z = \sum_{j=1}^{n} c_j x_j = c_1 x_1 + c_2 x_2 + ... + c_n x_n$ ...(25)

subject to m-constraints : ; $i = 1, 2, ..., m$ ...(26)

and satisfying the non-negativity restrictions : $x_j \geq 0$; $j = 1, 2, ..., n$ ...(27)

This form of the general linear programming problem is called its ***standard form*** and has the following characteristics :

(i) Objective function is either of maximization type or of minimization type

(ii) All constraints are in the form of equalities (equations) except for non-negativity restrictions which remain as inequalities ≥ 0.

(iii) Right hand side parameter of each equality constraint is non-negative.

(iv) All decision variables x_j; $j = 1, 2, ..., n$ are non-negative.

Presentation of a general linear programming problem into its ***standard form*** is often called ***reformulation of linear programming problem.***

The standard form of the linear programming problem is used to develop the procedure for solving general linear programming problem and finds its use in simplex method.

Reduction from One Form to Other Form of LPP

We generally need to convert (or reduce) the canonical form of LPP to its standard form and vice-versa in of its solution. Some important tips used in reduction from one form to other form of LPP are as follows :

1. Since the minimization of any function $f\,(x)$ is equivalent to the maximization of the negative expression of the function $f\,(x)$ i.e. min. [$f(x)$] = max. [– $f\,(x)$] = – max. [$f\,(x)$], therefore, the linear objective function min. Z = $c_1x_1 + c_2x_2 + \ldots + c_n\, x_n$ is equivalent to

$$\text{max. } Z' = \text{max. } (-Z) = -c_1x_1 - c_2x_2 - \ldots - c_n\, x_n$$

Consequently, the objective function can always be put in the maximization form in any linear programming problem.

2. Inequality constraints can always be converted to equalities by augmenting (adding or subtracting) the left hand sides of each such constraint by non-negative new variables. These new variable are called ***slack variables*** and are added if the constraints are of ≤ type or subtracted if the constraints are of ≥ type. Since, the subtracted variable represents the surplus of the left hand side over the right hand side, it is common to refer to it as ***surplus variable.*** For convenience, however, the name ***'slack variable'*** will also be used to represent this type of variable. In this respect, a surplus is regarded as a negative slack.

3. The right hand side parameter of an equality constraint can always be made positive by multiplying both sides of the constraint by – 1, whenever necessary.

4. So far, the decision variables $x_1, x_2, \ldots, x_n$ have been assumed to be all non-negative. In actual practice, these variables could also be zero or negative. If a variable is negative, it can always be expressed as the difference of two non-negative variables e.g. a negative variable x_r can be written as $x_r = x'_r - x''_r$, where $x'_r \geq 0$ and $x''_r \geq 0$. If the non-negativity restrictions on variables are not given, then the variables are said to be ***unrestricted in sign*** i.e. these may be positive, zero or negative and in this case; a variable which is unrestricted in sign is equivalent to the difference between two non-negative variables e.g. if x_r is a variable unrestricted in sign, then it can be replaced by $x'_r - x''_r$, where $x'_r \geq 0$ and $x''_r \geq 0$.

The above tips are useful while converting an LPP from any of the canonical forms to its standard form.

Also, it is not difficult to change the LPP from standard form to canonical form in as much as an equation $x = k$ is equivalent to two weak inequalities in opposite directions $x \leq k$ and $x \geq k$ i.e. $x \leq k$ and $-x \geq -k$. For example, the equality constraint $a_{11}x_1 + a_{12}x_2 + ... + a_{1n}\, x_n = b_1$ is equivalent to following two simultaneous linear constraints :

$$a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1 \text{ and } a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \geq b_1$$

or $\quad a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \leq b_1$ and $-a_{11}x_1 - a_{12}x_2 - ... - a_{1n}x_n \geq -b_1$.

The reduction from one form to other form of linear programming problems has been illustrated by following examples :

ILLUSTRATIVE EXAMPLES

Example 1. Reduce the following linear programming problem to its maximization canonical form :

Maximize Z = $2x_1 + 3x_2$, subject to $x_1 + x_2 = 5$, $5x_1 - 2x_2 \geq 3$, $x_1 \geq 0$, $x_2 \geq 0$.

Solution. Clearly, the given LPP is neither in the canonical form nor in the standard form. Since, we have to reduce (write) the given problem into its canonical form, therefore, we first replace the equality constraint $x_1 + x_2 = 5$ by two weak inequality constraints $x_1 + x_2 \leq 5$ and $x_1 + x_2 \geq 5$ in opposite directions.

Thus, the given LPP can be written as :

Maximize Z = $2x_1 + 3x_2$, subject to $x_1 + x_2 \leq 5$, $x_1 + x_2 \geq 5$, $5x_1 - 2x_2 \geq 3$ and $x_1 \geq 0$, $x_2 \geq 0$.

Again, the inequality constraints $x_1 + x_2 \geq 5$ and $5x_1 - 2x_2 \geq 3$ can be written as $-x_1 - x_2 \leq -5$ and $-5x_1 + 2x_2 \leq -3$ respectively. So, we can write the given LPP in the following canonical form :

Maximize Z = $2x_1 + 3x_2$ subject to $x_1 + x_2 \leq 5$, $-x_1 - x_2 \leq -5$, $-5x_1 + 2x_2 \leq -3$ and $x_1 \geq 0$, $x_2 \geq 0$.

Example 2. Bring in slack and surplus variables to write the following LPP in the standard form :

Maximize Z = $3x_1 - x_2$ subject to $x_1 - 2x_2 \leq -3$, $4x_1 + x_2 \leq 4$ and $x_1 \geq 0$, $x_2 \geq 0$.

Solution. The right hand side parameter in the constraint $x_1 - 2x_2 \leq -3$ can be made positive by multiplying both sides of it by -1 and so the inequality constraint $x_1 - 2x_2 \leq -3$ can be written as $-x_1 + 2x_2 \geq 3$. Therefore, we re-write the given LPP as follows :

Maximize Z = $3x_1 - x_2$ subject to $-x_1 + 2x_2 \geq 3$, $4x_1 + x_2 \leq 4$ and $x_1 \geq 0$, $x_2 \geq 0$.

Now, by introducing the surplus variable $s_1 \geq 0$ and the slack variable $s_2 \geq 0$, the inequality constraints $-x_1 + 2x_2 \geq 3$ and $4x_1 + x_2 \leq 4$ may be written respectively as follows :

$$-x_1 + 2x_2 - s_1 = 3 \text{ and } 4x_1 + x_2 + s_2 = 4.$$

Finally, the standard form of the given LPP is :

Maximize $Z = 3x_1 - x_2$ subject to $-x_1 + 2x_2 - s_1 = 3$, $4x_1 + x_2 + s_2 = 4$ and $x_1 \geq 0$, $x_2 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$, where s_1 is the surplus variable and s_2 is the slack variable.

Example 3. Write the following LPP into its standard form :

Maximize $Z = 3x_1 + 5x_2 + 7x_3$ subject to $6x_1 - 4x_2 \leq 5$, $3x_1 + 2x_2 + 5x_3 \geq 11$, $4x_1 + 3x_3 \leq 2$ and $x_1 \geq 0$, $x_2 \geq 2$.

Solution. Clearly, the decision variable x_3 is unrestricted in sign, therefore, we can replace x_3 by $x'_3 - x''_3$ i.e. $x_3 = x'_3 - x''_3$, where $x'_3 \geq 0$ and $x''_3 \geq 0$.

Thus, the given LPP can be written as :

Maximize $Z = 3x_1 + 5x_2 + 7x'_3 - 7x''_3$ subject to $6x_1 - 4x_2 \leq 5$, $3x_1 + 2x_2 + 5x'_3 - 5x''_3 \geq 11$, $4x_1 + 3x'_3 - 3x''_3 \leq 2$ and $x_1 \geq 0$, $x_2 \geq 0$, $x'_3 \geq 0$, $x''_3 \geq 0$.

Now, by introducing the slack variables $s_1 \geq 0$, $s_3 \geq 0$ and the surplus variable $s_2 \geq 0$, the inequality constraints may be written respectively as follows :

$$6x_1 - 4x_2 + s_1 = 5,\ 3x_1 + 2x_2 + 5x'_3 - 5x''_3 - s_2 = 11,\ 4x_1 + 3x'_3 - 3x''_3 + s_3 = 2.$$

Finally, the standard form of the given LPP is :

Maximize $Z = 3x_1 + 5x_2 + 7x'_3 - 7x''_3$ subject to $6x_1 - 4x_2 + s_1 = 5$, $3x_1 + 2x_2 + 5x'_3 - 5x''_3 - s_2 = 11$, $4x_1 + 3x'_3 - 3x''_3 + s_3 = 2$ and $x_1 \geq 0$, $x_2 \geq 0$, $x'_3 \geq 0$, $x''_3 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$, where s_1 and s_3 are the slack variables and s_2 is the surplus variable.

Example 4. Reformulate the following LPP into its standard form :

Minimize $Z = 2x_1 + x_2 + 4x_3$ subject to $-2x_1 + 4x_2 \leq 4$, $x_1 + 2x_2 + x_3 \geq 5$, $2x_1 + 3x_3 \leq 2$ and $x_1 \geq 0$, $x_2 \geq 0$.

Solution. Clearly, the decision variable x_3 is unrestricted in sign, therefore, we can express x_3 as $x_3 = x'_3 - x''_3$, where $x'_3 \geq 0$, $x''_3 \geq 0$.

Thus, the given LPP can be written as :

Minimize $Z = 2x_1 + x_2 + 4x'_3 - 4x''_3$, subject to $-2x_1 + 4x_2 \le 4$, $x_1 + 2x_2 + x'_3 - x''_3 \ge 5$, $2x_1 + 3x'_3 - 3x''_3 \le 2$ and $x_1 \ge 0$, $x_2 \ge 0$, $x'_3 \ge 0$, $x''_3 \ge 0$.

Now, by introducing the slack variables $s_1 \ge 0$, $s_3 \ge 0$ and the surplus variable $s_2 \ge 0$, the inequality constraints may be written respectively as follows :

$$-2x_1 + 4x_2 + s_1 = 4,\ x_1 + 2x_2 + x'_3 - x''_3 - s_2 = 5,\ 2x_1 + 3x'_3 - 3x''_3 + s_3 = 2$$

$\therefore$ The standard form of the given LPP is :

Minimize $Z = 2x_1 + x_2 + 4x'_3 - 4x''_3$, subject to $-2x_1 + 4x_2 + s_1 = 4$, $x_1 + 2x_2 + x'_3 - x''_3 - s_2 = 5$, $2x_1 + 3x'_3 - 3x''_3 + s_3 = 2$ and $x_1 \ge 0$, $x_2 \ge 0$, $x'_3 \ge 0$, $x'' \ge 0$, $s_1 \ge 0$, $s_2 \ge 0$, $s_3 \ge 0$, where s_1 and s_3 are the slack variables and s_2 is the surplus variable.

However, if the objective function of the above LPP is converted into maximization type, then it, is given as :

Maximize $Z' (= -Z) = -2x_1 - x_2 - 4x'_3 + 4x''_3$.

Finally, the standard form of the given LPP is :

Maximize $Z' (= -Z) = -2x_1 - x_2 - 4x'_3 + 4x''_3$, subject to $-2x_1 + 4x_2 + s_1 = 4$, $x_1 + 2x_2 + x'_3 - x''_3 - s_2 = 5$, $2x_1 + 3x'_3 - 3x''_3 + s_3 = 2$ and $x_1 \ge 0$, $x_2 \ge 0$, $x'_3 \ge 0$, $x''_3 \ge 0$, $s_1 \ge 0$, $s_2 \ge 0$, $s_3 \ge 0$, where s_1 and s_3 are the slack variables and s_2 is the surplus variable.

EXERCISE

1. Write the following LPP into its canonical form :

Maximize $Z = 2x_1 - 3x_2$ subject to $x_1 - x_2 \ge -3$, $2x_1 + 3x_2 \le 5$, $x_1 + 4x_2 \ge 4$ and $x_1 \ge 0$, $x_2 \ge 0$.

2. Reduce the following LPP to its canonical form :

Minimize $Z = x_1 - x_2 + x_3$ subject to $4x_1 + 2x_2 - x_3 \le -3$, $x_1 + 5x_2 + x_3 \ge 2$, $2x_1 - x_2 + 7x_3 \ge 4$ and $x_1 \ge 0$, $x_2 \ge 0$, $x_3 \ge 0$.

3. Reduce the following LPP to its canonical form :

Maximize $Z = x_1 + x_2$ subject to $x_1 - 3x_2 \le 5$, $2x_1 + x_2 \ge -2$ and $x_1 \ge 0$, x_2 is unrestricted in sign.

4. Express the following LPP in its canonical form :

Minimize $Z = 3x_1 + x_2$ subject to $x_1 + x_2 = 4$, $3x_1 - x_2 \le 1$and $x_1 \ge 0$, $x_2 \ge 0$.

5. Write the following LPP into its standard form :

Maximize $Z = 2x_1 + x_2 + 3x_3$ subject to $x_1 + 2x_2 + 3x_3 \leq 20, x_1 - 2x_2 + 5x_3 \geq 5, 2x_1 + 5x_2 - 7x_3 \geq 1$ and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

6. Reduce the following LPP to its standard form :

Maximize $Z = 3x_1 + 2x_2 + 5x_3$ subject to $-5x_1 + 2x_2 \leq 5, 2x_1 + 3x_2 + 4x_3 \geq 7, 2x_1 + 5x_3 \leq 3$ and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

7. Convert the following LPP to its standard form :

Maximize $Z = 3x_1 - 2x_2 + 4x_3$ subject to $x_1 + 2x_2 + x_3 \leq 8, \quad 2x_1 - x_2 + x_3 \geq 2, 4x_1 - 2x_2 - 3x_3 = -6$ and $x_1 \geq 0, x_2 \geq 0$.

8. Express the following LPP in its standard form :

Minimize $Z = 3x_1 + 4x_2$ subject to $2x_1 + 5x_2 \leq 3, 4x_1 - 3x_2 \geq 4$ and $x_1 \geq 0, x_2 \geq 0$.

9. Reformulate the following LPP into the standard form :

Minimize $Z = 2x_1 + x_2 + 4x_3$ subject to $-2x_1 + 4x_2 \leq 4, x_1 + 2x_2 + x_3 \geq 5, x_1 + 2x_2 + 3x_3 \leq 1$ and $x_1 \geq 0, x_2 \geq 0$.

ANSWERS

1. Maximize $Z = 2x_1 - 3x_2$ s.t. $-x_1 + x_2 \leq 3, 2x_1 + 3x_2 \leq 5, -x_1 - 4x_2 \leq -4$ and $x_1 \geq 0, x_2 \geq 0$.

2. Min. $Z = x_1 - x_2 + x_3$ s.t. $-4x_1 - 2x_2 + x_3 \geq 3, x_1 + 5x_2 + x_3 \geq 3, 2x_1 - x_2 + 7x_3 \geq 4$ and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$.

3. Maximize $Z = x_1 + x'_2 - x''_2$ s.t $.x_1 - 3x'_2 + 3x''_2 \leq 5, -2x_1 - x'_2 + x''_2 \leq 2$ and $x_1 \geq 0, x'_2 \geq 0, x''_2 \geq 0$, where $x'_2 - x''_2 = x_2$.

4. Min. $Z = 3x_1 + x_2$ s.t. $x_1 + x_2 \geq 4, -x_1 - x_2 \geq -4, -3x_1 + x_2 \geq -1$ and $x_1 \geq 0, x_2 \geq 0$.

5. Maximize $Z = 2x_1 + x_2 + 3x_3$ s.t. $x_1 + 2x_2 + 3x_3 + s_1 = 20, x_1 - 2x_2 + 5x_3 - s_2 = 5, 2x_1 + 5x_2 - 7x_3 - s_3 = 1$ and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$, where s_1 is the slack variable and s_2, s_3 are the surplus variables.

6. Maximize $Z = 3x_1 + 2x_2 + 5x_3$ s.t. $-5x_1 + 2x_2 + s_1 = 5, 2x_1 + 3x_2 + 4x_3 - s_2 = 7, 2x_1 + 5x_3 + s_3 = 3$ and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0$, where s_1 and s_3 are the slack variables and s_2 is the surplus variable.

7. Maximize $Z = 3x_1 - 2x_2 + 4x'_3 - 4x''_3$ s.t. $x_1 + 2x_2 + x'_3 - x''_3 + s_1 = 8, 2x_1 - x_2 + x'_3 - x''_3 - s_2 = 2, -4x_1 + 2x_2 + 3x'_3 - 3x''_3 = 6$ and $x_1 \geq 0, x_2 \geq 0,$

$x'_3 \geq 0$, $x''_3 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$, where s_1 is the slack variable and s_2 is the surplus variable and $x'_3 - x''_3 = x_3$.

8. Maximize $Z' (= -Z) = -3x_1 - 4x_2$ s.t. $2x_1 + 5x_2 + s_1 = 3$, $4x_1 - 3x_2 - s_2 = 4$ and $x_1 \geq 0$, $x_2 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$, where s_1 is the slack variable and s_2 is the surplus variable.

9. Maximize $Z' (= -Z) = -2x_1 - x_2 - 4x'_3 + 4x''_3$ s.t. $-2x_1 + 4x_2 + s_1 = 4$, $x_1 + 2x_2 + x'_3 - x''_3 - s_2 = 5$, $x_1 + 2x_2 + 3x'_3 - 3x''_3 + s_3 = 1$ and $x_1 \geq 0$, $x_2 \geq 0$, $x'_3 \geq 0$, $s_1 \geq 0$, $s_2 \geq 0$, $s_3 \geq 0$, where s_1 and s_3 are the slack variables and s_2 is the surplus variable and $x'_3 - x''_3 = x_3$.

MODEL REVIEW QUESTIONS

1. What do you mean by quantitative analysis ?

2. What do you mean by linear programming problem ? Why it is called so?

3. State clearly the basic assumptions that are made in linear programming problem.

4. What are the limitations of linear programming technique ?

5. Give a brief account of applications of linear programming.

6. Explain the meaning of LPP stating its uses and give its limitations.

7. What are the advantages of linear programming technique ?

8. Define simple and general linear programming problems and give their examples.

9. Write note on problem solving and decision making ?

10. What do you mean by canonical form of a linear programming problem ? Illustrate it by giving an example.

11. What do you mean by standard form of a linear programming problem ? Illustrate it by giving an example.

12. Give the characteristics of the standard form of a linear programming problem.

13. Give the characteristics of the canonical form of a linear programming problem.

14. Compare the canonical and standard forms of a linear proramming problem.

15. Define slack and surplus variables as involved in the LPP. How are these variables useful in solving an LPP ?

MULTIPLE CHOICE QUESTIONS ON LINEAR PROGRAMMING

1. Mathematical model of a linear programming problem is important because

(a) it helps in conversing the verbal description and numerical data into mathematical expression.

(b) decision-makers prefer to work with formal models.

(c) it captures the relevant relationship among decision factors.

(d) it enables the use of algebraic technique.

2. Linear programming is a

(a) constrained optimization technique.

(b) technique for economic allocation of limited resources.

(c) mathematical technique

(d) all of the above.

3. A constraint in a linear programming model restricts

(a) value of objective function **(b)** value of a decision variable.

(c) use of the available resource. **(d)** all of the above

4. The distinguishing feature of a linear programming model is

(a) relationship among all variables is linear

(b) it has single objective function and constraints

(c) value of decision variables is non-negative

(d) all of the above

5. Constraints in a linear programming model represent

(a) limitations **(b)** requirements

(c) balancing limitations and requirements **(d)** all of the above

6. Non-negativity condition is an important component of a linear programming model because

(a) variables value should remain under the control of decision-maker.

(b) value of variables make sense and correspond to real-world problems.

(c) variables are interrelated interms of limited resources.

(d) none of the above.

7. Before formulating a formal linear programming model, it is better to

(a) express each constraint in words.

(b) express the objective function in words.

(c) decision variables are identified verbally

(d) all of the above.

8. Each constraint in a linear programming model is expressed as an

(a) inequality with ≥ sign **(b)** inequality with ≤ sign

(c) equation with = sign **(d)** none of the above.

9. Maximization of objective function in a linear programming model means.

(a) value occurs at allowable set of decisions

(b) highest value is chosen among allowable decisions

(c) neither of above **(d)** both (a) and (b).

10. Which of the following is not a characteristic of a linear programming model ?

(a) alternative courses of action

(b) an objective function of maximization type

(c) limited amount of resources

(d) non-negativity condition on the value of decision variables.

11. Which of the following conditions are necessary for applying linear programming ?

(1) There must be a well defined objective function

(2) The decision variables should be interrelated and non-negative.

(3) The resources must be in limited supply.

(a) (1) and (2) only **(b)** (1) and (3) only

(c) (2) and (3) only **(d)** (1), (2) and (3)

12. Consider the following statements regarding the characteristics of the standard form of a linear programming.

(1) All the constraints are expressed in the form of equations.

(2) The right hand side of each constraint equation is non-negative.

(3) All the decision variables are non-negative.

Which of the following statements are correct ?

(a) (1), (2) and (3) **(b)** (1) and (2)

(c) (2) and (3) **(d)** (1) and (3)

13. The conversion of the constraint $2x_1 + 3x_2 \leq 15$ in equation form with the slack variable $s \geq 0$ will be

(a) $2x_1 + 3x_2 + s = 15$ **(b)** $2x_1 + 3x_2 - s = 15$

(c) $2x_1 + 3x_2 \pm s = 15$ **(d)** none of the above

14. Which of the following is not a linear programming problem.

(a) Max. $Z = 2x_1 + 5x_2$

s.t. $2x_1 + 3x_2 \leq 5,$

$3x_1 - 4x_2 \geq 6$

and $x_1 \geq 0, x_2 \geq 0$

(b) Min. $Z = 3x_1 - 4x_2$

s.t. $3x_1 + 4x_2 \leq -2,$

$4x_1 - 5x_2 = 6$

and $x_1 \geq 0, x_2 \geq 0$

(c) Max. $Z = 2x_1 - 7x_2$

s.t. $3x_1 + 5x_2 = 8,$

$5x_1 - 2x_2 = 6$

and $x_1 \geq 0, x_2 \geq 0$

(d) Max. $Z = 5x_1 + 3x_2$

s.t. $3x_1 + x_2^2 = 1,$

$5x_1 - 2x_2 = 3$

and $x_1 \geq 0, x_2 \geq 0.$

15. Consider the following statements :

Linear programming model can be applied to (1) Line balancing problem (2) transportation problem (3) project management. Which of the following is correct

(a) 1, 2 and 3 **(b)** 1 and 2

(c) 2 and 3 **(d)** 1 and 3.

ANSWERS

1. (a) **2.** (d) **3.** (d) **4.** (a) **5.** (d) **6.** (b) **7.** (d) **8.** (d)

9. (a) **10.** (b) **11.** (d) **12.** (a) **13.** (a) **14.** (d) **15.** (d)

EXERCISE ON LPP FOR PRACTICE

1. Identify the following problems as LPP or non-LPP ? Also, give reason.

(a) Maximize $Z = 2x_1 + 3x_2$

subject to $x_1 - x_2 \geq 5$,

$2x_1 - 3x_2 \geq -1$

and $x_1 \geq 0, x_2 \geq 0$

(b) Maximize $Z = -x_2$

subject to $x_1 - x_2 \geq 2$

$2x_1 - 5x_2 \leq 4$

and $x_1 \geq 0, x_2 \geq 0$

(c) Minimize $Z = 2x_1 - 3x_2$

subject to $x_1^2 + x_2^2 \leq 7$,

$2x_1 - 5x_2 \leq 1$

and $x_1 \geq 0, x_2 \geq 0$

(d) Maximize $Z = 5x_1 + 4x_2$

subject to $3x_1 + 2x_2 \geq 5$,

$x_1 - 2x_2 \leq 2$

and $x_1 \geq 0, x_2 \geq 0$.

2. Food X contains 6 units of vitamin A and 7 units of vitamin B per gram and costs `12 per gram. Food Y contains 8 units and 12 units of vitamins A and B per gram respectively and costs `20 per gram. The daily requirements of vitamins A and B are at least 100 units and 120 units respectively.

Formulate the above as a linear programming problem.

3. A company produces two types of leather belts, say, type A and type B. Belt A is of superior quality and belt B is of inferior quality. Profits on the two types of belts are ` 4 and ` 3 per belt respectively. Each belt of type A requires twice as much time as that for a belt of type B. If all belts were of type B, the company could produce 1000 belts per day. But the supply of leather is sufficient only for 800 belts per day. Belt A requires a fancy buckle and only 400 buckles are available for this per day. For belt of type B, only 700 buckles are available per day. How should the company will manufacture the two types of belts so that the profit is maximum ?

Formulate the above as a linear programming problem.

4. Write the following LPP in its canonical form :

Maximize $Z = x_1 - 2x_2$

subject to $x_1 + x_2 = 2,\ 2x_1 - x_2 \leq -1$

and $x_1 \geq 0$, x_2 is unrestricted.

5. Convert the following LPP into its standard form :

Maximize $Z = x_1 - 2x_2 + 5x_3 + x_4$

subject to $3x_1 + 4x_2 - 3x_3 + 7x_4 \leq 2,$

$2x_1 - 3x_2 + x_3 + 5x_4 \geq 5,$

$2x_1 - 7x_2 + 5x_3 + x_4 \geq 3$

$x_1 + x_2 - 3x_3 - 4x_4 \geq -1$

and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4$ is unrestricted.

6. Which of the following LPP are in canonical form and which are in standard form ?

(a) Maximize $Z = 4x_1 + 3x_2$

subject to $5x_1 - 3x_2 \leq 5,$

$3x_1 - 2x_2 \leq 1$

and $x_1 \geq 0, x_2 \geq 0$

(b) Minimize $Z = 2x_1 - x_2$

subject to $2x_1 + x_2 \geq 1,$

$3x_1 + 5x_2 \geq 4$

and $x_1 \geq 0, x_2 \geq 0$

(c) Maximize $Z = 5x_1 - 2x_2$

subject to $3x_1 - 4x_2 = 5,$

$2x_1 + 3x_2 = 7$

and $x_1 \geq 0, x_2 \geq 0$

(d) Minimize $Z = 2x_1 - 7x_2$

subject to $x_1 - 2x_2 = 3,$

$x_1 + 3x_2 = 7$

and $x_1 \geq 0, x_2 \geq 0.$

ANSWERS

1(a) This is an LPP, as the objective function and all the constraints are linear.

(b) This is not an LPP as the objective function is not linear but quadratic. This is infact a quadratic programming problem.

(c) This is not an LPP as one of the constraints is not linear.

(d) This is an LPP as the objective function and all the constraints are linear.

2. Minimize $Z = 12x_1 + 20x_2$ subject to $6x_1 + 8x_2 \geq 100$, $7x_1 + 12x_2 \geq 120$ and $x_1 \geq 0, x_2 \geq 0$

3. Maximize $Z = 4x_1 + 3x_2$ subject to $2x_1 + x_2 \leq 1000$, $x_1 + x_2 \leq 800$, $x_1 \leq 400$, $x_2 \leq 700$ and $x_1 \geq 0, x_2 \geq 0$.

4. Maximize $Z = x_1 - 2x_2$ subject to $x_1 + x_2' - x_2'' \leq 2$, $-x_1 - x_2' + x_2'' \leq -2$, $2x_1 - x_2' + x_2'' \leq -1$ and $x_1 \geq 0, x_2' \geq 0, x_2'' \geq 0$, where $x_2' - x_2'' = x_2$.

5. Maximize $Z = x_1 - 2x_2 + 5x_3 + x_4' - x_4''$ subject to $3x_1 + 4x_2 - 3x_3 + 7x_4' - 7x_4'' + s_1 = 2, 2x_1\ 3x_2 + x_3 + 5x_4' - 5x_4'' - s_2 = 5, 2x_1 7x_2 +$

$5x_3 + x_4' - x_4'' - s_3 = 3, -x_1 - x_2 + 3x_3 + 4x_4'' - 4x_4'' + s_4 = 1$ and $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4' \geq 0, x_4'' \geq 0, s_1 \geq 0, s_2 \geq 0, s_3 \geq 0, s_4 \geq 0,$ where s_1 and s_4 are the slack variables and s_2 and s_3 are the surplus variables and $x_4' - x_4'' = x_4$.

6. **(a)** Canonical form **(b)** Canonical form

(c) Standard form **(d)** Standard form.